

Modifications to the behavior of a circular hydraulic jump in the presence of a soluble surfactant

Paul Stanley⁺ and Xiyu Du

Department of Physics and Astronomy, Beloit College, Beloit, Wisconsin USA

Abstract. The circular hydraulic jump in the presence of a soluble surfactant can display unexpected static ripple features in the inner super-critical flow region. These ripples are likely capillary surface waves pushed inward from the region of the hydraulic jump by surface tension gradient Marangoni stresses, therefore able to exist in a region thought to be without surface features. The additional surface drag requires modifications to the Watson theory of circular hydraulic jumps in order to predict jump location and inner ripple radius. We have developed a first order correction to the theory that qualitatively explains the surfactant concentration on the location of the jump and the inner ripple ring; the theory shows reasonable agreement with experiment.

Keywords: hydraulic jump, surfactant, Marangoni stress, surface tension

1. Introduction

The theory of the circular hydraulic jump is well studied. Raleigh[1] did early work on inviscid models, Watson[2] considered the effects of viscosity, while Bush and Aristoff[3] provided corrections for surface tension. Others have attempted to model the circular hydraulic jump in terms of simpler scaling arguments, a comprehensive study of which can be found in Khavari, et. al.[4] Though the fundamental problem might be considered to be effectively understood, there are still unresolved issues. Much of the focus since the late 1990's has been on the formation of stable polygonal structures for certain parameters of the fluid flow rate and viscosity[5].

The original intention of this experiment was to develop careful measurements of the radius of the hydraulic jump as a function of flow rate, and possibly include this experiment as one of the intermediate physics lab exercises at the College. One of the first undergraduate students working on the equipment design found it challenging to completely wet the surface of the Plexiglas disk, and tried adding soap solution. He noticed a difficult to resolve inner ring, and reported it at the Midstates Consortia Undergraduate Research Symposium[6]. After improving the equipment, it was possible to resolve the inner ring more clearly, as in Fig. 1.

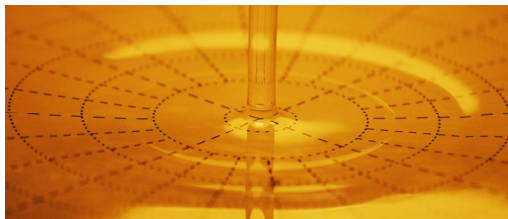


Fig. 1: Dotted circles are at 0.5 cm, 1.5 cm, 2.5 cm, etc. The hydraulic jump is at 3 cm and the Marangoni ring/ripples at 2 cm, which is well inside the super critical region.

⁺ Corresponding author. Tel.: +16083632226; fax: +16083632052.
E-mail address: stanleyp@beloit.edu.

With additional measurements, it was concluded that the effect was a result of the surfactant, and not simply the reduced surface tension or of any interactions between the fluid mixture and the horizontal impingement Plexiglas surface. This inner feature was originally thought to be a weak hydraulic jump, which would violate naive models of energy conservation in a hydraulic jump, since the flow both before and after the jump was super-critical. It is now thought that these ripples are Marangoni stress induced ripples, with a consequence that much of the Watson theory requires modifications. Though Bush and Aristoff point out that surface tension gradients will complicate results, neither they nor any other group appear to have noticed these ripples.

2. Theory

The original Watson[2] paper was based on the assumption that the fluid flow consisted of radial velocity $u(r, z)$ and vertical velocity $w(r, z)$, and that there was no angular dependence to the problem. We outline the Watson approach below. The goal was to find fluid depth $h(r)$ as well as the velocity profile. The flow rate would be given by

$$Q = 2\pi r \int_0^{h(r)} u dz.$$

By direct application of equations for conservation of fluid flow

$$\partial(ru) / \partial r + \partial(rw) / \partial z = 0,$$

the Navier-Stokes Equation

$$u(\partial u / \partial r) + w(\partial u / \partial z) = \nu(\partial^2 u / \partial z^2),$$

and boundary conditions for the top and bottom surfaces of the circular outflow region

$$u = w = 0 \text{ at } z = 0 \text{ and } \partial u / \partial z = 0 \text{ at } z = h(r), \quad (1)$$

and assuming a self-similar solution of the form $u(r, z) = U(r)f(\eta)$ where $\eta = z / h$, Watson was able to find unique solutions for $U(r)$ and $h(r)$ given by

$$U(r) = \frac{27c^2}{8\pi^4} \frac{Q^2}{\nu(r^3 + l^3)} \text{ and } h(r) = \frac{2\pi^2}{3\sqrt{3}} \frac{\nu(r^3 + l^3)}{Qr}, \quad (2)$$

with c a numerical constant, and l is determined by boundary layer growth. The appropriate $f(\eta)$ was of the form of a Jacobian elliptical function. Henceforth, we will write u_w as the Watson solution.

The condition for a hydraulic jump is found by selecting a control volume and balancing forces against momentum flux

$$\frac{1}{2} \rho g w h_2^2 - \frac{1}{2} \rho g w h_1^2 = \rho w \int_0^{h_1} u_1^2 dy - \rho w \int_0^{h_2} u_2^2 dy$$

Watson ignored surface tension; Bush & Aristoff[3] include it, the result is then an expression for the jump location

$$\frac{r_j d^2 g a^2}{Q^2} \left(1 + \frac{2}{B_0} \right) + \frac{a^2}{2\pi r_j d} = \frac{0.01676}{\left(\frac{r_j}{a} \right)^3 \frac{1}{Re} + 0.1826} \quad (3)$$

Surface tension effects (at the location of the jump), are determined by the Bond number,

$$B_0 = \frac{\rho g r_j (d - h(r_j))}{\sigma}$$

At this point we diverge from Watson with additional physics. Our approach is to observe that the flow divides into two parts in the region after the ripples, where $r_j > r > r_i$. The boundary conditions of Equation (1) are modified such that

$$u = w = 0 \text{ at } z = 0, z = h_s(r) \text{ and } \partial u / \partial z = 0 \text{ at } z = h_t(r)$$

where h_s is the new surface height, and h_t is the height at which $\partial u / \partial z = 0$. For fully developed flow, there will be no mass flux or momentum flux across the h_t boundary. We assume that this restriction for no

mass flux or momentum flux also occurs during the boundary layer growth off of the top surface as well as for the fully developed region. As such, the new self-similar solution for lower and upper flow is given by

$$u_l = U_l(r) f\left(\frac{z}{h_l}\right) \text{ if } z < h_l \text{ and } u_u = U_u(r) f\left(\frac{h_s - z}{h_s - h_l}\right) \text{ if } z > h_l$$

where U_l is the fluid velocity along the curve defined by $z = h_l$. The form of $f(\eta)$ is identical to the Watson solution. Conservation of fluid flow requires

$$Q = 2\pi r \int_0^{h_l(r)} u_l dz + 2\pi r \int_{h_l(r)}^{h_s(r)} u_u dz.$$

Combining, we find $h_l h_s = h_w^2$, where h_w is the original Watson surface height. We also require that the boundary layer grows off the top surface in such a way as to not affect the behaviour at the bottom surface with respect to the original Watson flow u_w , so that

$$\partial u_w / \partial z |_{z=0} = \partial u_l / \partial z |_{z=0}$$

with immediate consequence $U_l = U_w h_l / h_w$.

Finally, we define the height layer h_x to be the continuation of the h_l transition line for fully developed flow. By symmetry, $h_s = 2h_x$. Then

$$h_s = \sqrt{2} h_w \text{ and } U_x = U_w / \sqrt{2} \quad (4)$$

with U_w and h_w defined by the original Watson equations, Equations (2). Most of these features are illustrated in Fig. 2.

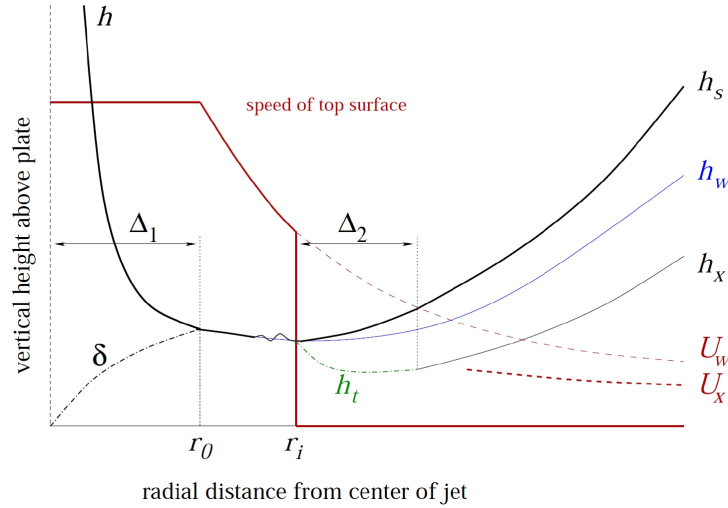


Fig. 2: Flow depth and velocity profiles for original Watson and modified flow in presence of surfactant.

Once the hydraulic jump is located it is possible to consider the location of the inner rings. To do this we balance the Marangoni stresses associated with the varied surface tension against the viscous drag on the top layer, following the example as presented for fluid pipe by Hancock and Bush[7]:

$$2\pi r_i \Delta\sigma = \pi \int_{r_i}^{r_j} \rho \nu \frac{\partial u}{\partial y} \Big|_{z=h_s} r dr$$

Though difficult to integrate, the solution will be of the form

$$2\pi r_i \Delta\sigma \approx \pi (r_j + r_i) (r_j - r_i) \rho \nu u_c / y_c,$$

where u_c is some characteristic speed, and y_c some characteristic depth in the region between the inner ripple and the jump. It is reasonable to assume that $y_c \approx h_x$ and $u_c \approx U_x$, as found in Equation (4). These quantities are not constant, but an order of magnitude estimate can be obtained by selecting the minimum h and maximum U , which are on the order of $100\mu\text{ m}$ and 1.0 m/s , respectively.

A more difficult estimate comes from trying to determine $\Delta\sigma$, the change in surface tension at the location of the inner ripples. We can start with a measure of surface tension in the presence of a surfactant[8]

$$\sigma = \sigma_0 - RT(c - c_b)h / 2$$

where R is ideal gas constant, T is temperature, c is volume concentration of surfactant, c_b is bulk concentration, and h the fluid thickness. h changes rapidly from ≈ 2 mm to $\approx 200\mu\text{m}$ on a time scale much faster than the diffusion of the surfactant, so the bulk concentration is effectively constant. As such, the correction to the surface tension before the ripples is at least a factor of ten smaller than the correction of the surface tension after the ripples. We can then assume that

$$\Delta\sigma = RT(c - c_b)h / 2 = \frac{RT}{\omega} \ln(1 + K_{ad}c),$$

with ω and K_{ad} constants, an expression known as the Szyszkowski equation. For sufficiently large concentration, we then expected

$$r_j - r_i \approx \frac{U_x}{h_x} \frac{1}{\rho\nu} \frac{RT}{\omega} \ln(1 + K_{ad}c) \approx A + B \ln c \quad (5)$$

with A and B constants.

3. Experiment

A 30.5 cm long stainless steel tube with a 0.72 cm inner diameter and a 0.92 cm outer diameter was used to project a stream of water vertically downward onto a horizontal flat circular disk of acrylic Plexiglas. The edges of the disk were rounded to allow the water to flow smoothly over the sides. Originally large diameter neoprene O-rings of various thicknesses were to be used to control water depth far from the point of impingement, but it was found that a relatively constant depth could be maintained without the rings. The disk was supported by three adjustable screws threaded in an aluminium disk for levelling. The position of the disk beneath the source tube could be varied between 0 cm and 25 cm. The shortest distance that was used for data collection was 5 cm.

Water was supplied to the source tube through a meter of flexible hose with 1.3 cm inner diameter connected to a large 25 litre reservoir of water positioned so that the height of the top water surface was 95 cm above the disk when the distance between the disk and the stainless steel tube was 10.9 cm. A needle valve was connected to the other end of the hose and the valve and the stainless tube was connected by another meter of flexible hose with 0.92 cm inner diameter. The valve is used to control the flow rate in a more exact fashion. The water head height was controlled via an overflow tube from the reservoir. A return pump recycled the water when the concentration of the soap solution needed to remain the same.

In general there were two different procedures of making the measurements depended on which feature was unchanged. The easier procedure was keeping the concentration of the soap solution constant. In this procedure the needle valve was first turned to the maximum and then adjusted to where the water flow was stable. At this point, the radius of the jump was about 3.5 cm to 4 cm. Then a measuring cup was used to hold water and a stop watch was used to measure how long it took to fill the cup. After that, a graduated cylinder was used to measure the volume of the water that was in the cup. This process was repeated five times in order to eliminate the error while measuring the flow rate. The radius of the jump and the inner ring was roughly measured by a ruler for future reference and a picture was taken to make more exact measurements of the radii. The soap solution that was left in the container would be pumped back to the reservoir in order to keep the same concentration. The next step was adjusting the needle valve to obtain a smaller flow rate. An elimination of 0.3 cm in the outer jump was a reasonable step size. Repeat the whole process until the radius of the outer jump reached about 0.7 cm.

The other procedure was more time consuming. The general step was still the same as the former one with two things different. First, the needle valve will be adjusted at the very beginning such that it could form a jump that was around 3 cm and after that the needle valve would remain unchanged in order to keep a constant flow rate. Second, the water would not be recycled after every set of data collection but would be pumped out of the system. Fresh water would be added after each run with an increasing amount of soap having a step size of 5 mL. The reason for refreshing water after every set of data was to reduce the amount of foam. A huge amount of foam would result in a decrease in soap concentration and thus reduce the accuracy of data.

After all the data were collected, they would be entered in a spreadsheet for various calculations. A program called PixelStick was used to measure the radius of the jump and inner ring directly off of the digital photographs. The way it worked was when taking the pictures in lab; a sheet of plastic paper was placed on the disk with concentric circles that varied 1 cm in radius with each other. By using PixelStick, how many pixels was 1 cm could be determined and then used as a ruler for measuring the diameter to within 5 % for the down-flow, and within 0.5 % for the ring radii.

Water depth in the super-critical region of the hydraulic jump was measured by a 1950's era machine shop depth gauge, with an accuracy of 1/2000 of an inch.

Flow rates were kept below the level of turbulence in the pipe; in general, the Reynold's number Re was less than 8000. As such, the pipe flow was assumed to be Poisselle in nature. The distance between the pipe outlet and the surface was usually kept fixed at about 11 cm, and it was expected that the fluid flow from the pipe had transitioned to uniform flow by then.

4. Results

The results for the hydraulic jump radius r_j , location of inner ripples r_i , and difference $r_j - r_i$ for varying soap concentration are shown in Fig. 3. The slight outward shift in the location of the hydraulic jump is expected by the correction for surface tension according to Bush; the data for pure water is somewhat unreliable, owing to bubbles which would accumulate in the source tubing and difficulties in wetting the surface. The difference between the jump radius and inner ripples, $r_j - r_i$, agrees well with a logarithmic dependence on soap concentration, as expected from Equation (5).

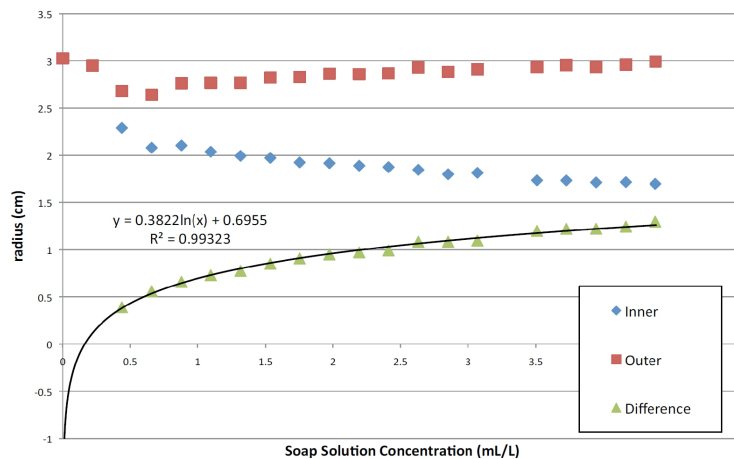


Fig. 3: Hydraulic jump radius r_j , location of inner ripples r_i , and difference $r_j - r_i$ for varying soap concentration.

The hydraulic jump locations in this flow regime agrees fairly well with the correction for surface dragging, as shown in Fig. 4, though the results are still off from the modified theory by an amount on order of the same 50 % as the previous graph.

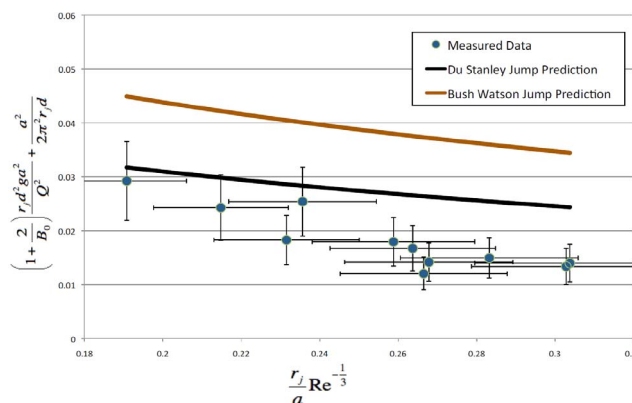


Fig 4: Hydraulic jump location predictions compared with actual measurements. This is a plot of the left hand side of Eq. (3) on the vertical axis against the dimensionless parameter $r_j / aRe^{1/3}$

The fluid depth does grow rapidly after the inner ripples rings, as shown in Fig. 5. The depth is some 50 % deeper than even the correction, implying that the model that ignores the mass and momentum transfer during the boundary layer growth is insufficient to explain the full phenomena. That missing 50 % is likely to affect the correction to the jump radius location, as the deeper water would mean that the jump would shift because of increased interior pressure, and slower flow.

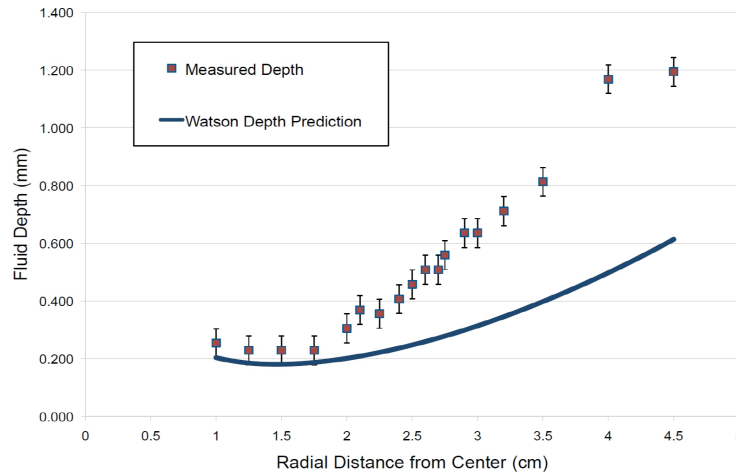


Fig. 5: Depth measurements, compared with Watson formula predictions for h_w . The significant increase for $r > 2$ cm corresponds to the region between the Marangoni ripples and hydraulic jump. The hydraulic jump is actually visible in this data, and occurs at about $r = 4$ cm.

Continued work is needed. Measurements of surface tension are very difficult, and the theory of soluble surfactants is complicated. Comparison with other surfactants, particularly pure Sodium Dodecyl Sulphate as opposed to industrial dish detergent, will occur soon. Finally, it is necessary to properly deal with the boundary layer growth and the integrals for the surface tension forces.

5. References

- [1] L. Rayleigh. On the theory of long waves and bores. *Proc. Roy. Soc. Lond. A.*, 90:324, 1914.
- [2] E. J. Watson. The radial spread of a liquid jet over a horizontal plane. *Journal of Fluid Mech.*, 20:481–499, 1964.
- [3] J. W. M. Bush and J. M. Aristoff. The influence of surface tension on the circular hydraulic jump. *Journal of Fluid Mech.*, 489:229–238, 2003.
- [4] Mohammad Khavari, Mohammad Passandideh-Fard, and Ali Reza Teymourtash. A Parametric Study on the Formation of a Circular Hydraulic Jump by Volume of Fluid Method. *The 6th International Chemical Engineering Congress & Exhibition*, Nov. 2009.
- [5] Ellegard, C., Hansen, A., Haaning, A., Hansen, K., Marcusson, A., Bohr, T., Lundbeck-Hansen, J., Watanabe, S., Creating Corners in Kitchen Sinks. *Nature*, 392, 767, 1998
- [6] Michael Mandel. Circular hydraulic jump. *Undergraduate Research Symposium - Physical Sciences, Mathematics and Computer Science*, University of Chicago, 2009. Midstates Consortium for Math and Science. Poster P2-6.
- [7] Matthew Hancock and John W. M. Bush. Fluid pipes. *Journal of Fluid Mech.*, 466:285–304, 2002.
- [8] Tuan Tran, Pinaki Chakraborty, Gustavo Gioia, Stanley Seers, and Wallace Goldburg. Marangoni shocks in unobstructed soap-film flows. *arXiv*, 2009.